COMMUTATORS OF TRACE ZERO MATRICES OVER PRINCIPAL IDEAL RINGS

ALEXANDER STASINSKI

ABSTRACT. We prove that for every trace zero square matrix A of size at least 3 over a principal ideal ring R, there exist trace zero matrices X, Y over R such that $XY - YX = A$. Moreover, we show that X can be taken to be regular mod every maximal ideal of R. This strengthens our earlier result that A is a commutator of two matrices (not necessarily of trace zero), and in addition, the present proof is simpler than the earlier one.

1. INTRODUCTION

Let R be a principal ideal ring, which we will always take to be commutative with identity (e.g., R could be a field). We let $\mathfrak{gl}_n(R)$ denote the Lie algebra of $n \times n$ matrices over R with Lie bracket $[X, Y] = XY - YX$, and $\mathfrak{sl}_n(R)$ the sub Lie algebra of trace zero matrices. In case $R = K$ is a field, a theorem of Albert and Muckenhoupt [1] says that every $A \in \mathfrak{sl}_n(K)$ is a commutator in $\mathfrak{gl}_n(K)$, that is, there exist $X, Y \in \mathfrak{gl}_n(K)$ such that $[X, Y] = A$. To go beyond the field case requires new ideas and the first major step was taken by Laffey and Reams [4] who proved the analogous result for $R = \mathbb{Z}$, solving a problem posed by Vaserstein [8, Section 5]. Whether every element in $\mathfrak{sl}_n(R)$ is a commutator in $\mathfrak{gl}_n(R)$ for a PIR R, was an open problem going back implicitly at least to Lissner [5], and was settled in the affirmative in [6].

In light of the above results, a natural question is whether X and Y can be taken in $\mathfrak{sl}_n(R)$, rather than just $\mathfrak{gl}_n(R)$. When $R = K$ is a field, it is known by work of Thompson [7, Theorems 1-4] that any $A \in \mathfrak{sl}_n(K)$ can be written as $A = [X, Y]$ for some $X, Y \in \mathfrak{sl}_n(K)$, except when $\operatorname{char} K = 2$ and $n = 2$. A generalisation of Thompson's result, allowing X and Y to lie in an arbitrary hyperplane in $\mathfrak{gl}_n(K)$ (but assuming $n > 2$ and $|K| > 3$), was recently obtained by de Seguins Pazzis [2]. On the other hand, it does not seem possible to modify our proof in [6] to yield the stronger assertion that every $A \in \mathfrak{sl}_n(R)$, with $n \geq 3$, is a commutator of matrices in $\mathfrak{sl}_n(R)$, even in the case where R is a field.

The main result of the present paper is that for any principal ideal domain (henceforth PID) R and $A \in \mathfrak{sl}_n(R)$, with $n \geq 3$, there exist $X, Y \in \mathfrak{sl}_n(R)$ such that $A = [X, Y]$. It is also easy to see that when 2 is invertible in R, the same conclusion holds for $A \in \mathfrak{sl}_2(R)$. Moreover, it follows from our proof that X can be chosen to be regular mod every maximal ideal of R (this was stated as an open problem in [6]). Our proof is significantly simpler than the proof of the main result in [6], and the new idea is to consider the matrices

$$X(\mathbf{x},a)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ x_1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & 0 & \vdots & \ddots & 1 \\ x_{n-1} & a & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{sl}_n(R),$$

where $\mathbf{x}=(x_1,\dots,x_{n-1})^{\mathsf{T}}\in R^{n-1}$ and $a\in R$; see Section 3. These matrices have some remarkable properties which let us carry through the proof. More precisely, we show that for a given non-scalar $A\in\mathfrak{sl}_n(R)$ in Laffey–Reams form (see [6, Theorem 5.6]), we can find $\mathbf{x}$ and a such that

$$\operatorname{tr}(X(\mathbf{x},a)^r A)=0, \quad \text{for } r=1,\dots,n-1,$$

and at the same time ensure that $X(\mathbf{x},a)$ mod $\mathfrak{p}$ is regular in $\mathfrak{gl}_n(R/\mathfrak{p})$, for every maximal ideal $\mathfrak{p}$ of R, as well as regular in $\mathfrak{sl}_n(R/\mathfrak{p})$, for any $\mathfrak{p}$ for which A is non-scalar mod $\mathfrak{p}$. We note that the condition on the vanishing of traces above is rather delicate, given that we also want $X(\mathbf{x},a)$ to have the above regularity property and trace zero, and depends on the existence of a solution of a system of polynomial equations over R, which in most cases is hopelessly complicated. Nevertheless, for the matrices $X(\mathbf{x},a)$ the system of equations becomes atypically simple, and we are able to show that a solution exists. We then use the well known local-global principle for systems of linear equations over rings, applied to the system defined by $[X(\mathbf{x},a),Y]=A$, $Y\in\mathfrak{sl}_n(R)$. Working over the localisation $R_{\mathfrak{p}}$ at a maximal ideal $\mathfrak{p}$ of R, we use a variant of the criterion of Laffey and Reams (see Section 2, Proposition 2.4) to show that the system has a solution if A is non-scalar mod $\mathfrak{p}$. Here we use that A mod $\mathfrak{p}$ is not merely regular in $\mathfrak{gl}_n(R/\mathfrak{p})$ but also regular in $\mathfrak{sl}_n(R/\mathfrak{p})$. The existence of a solution over $R_{\mathfrak{p}}$ when $\mathfrak{p}$ is such that A mod $\mathfrak{p}$ is scalar is more subtle and requires a separate argument. The existence of a local solution for every maximal ideal $\mathfrak{p}$ then implies the existence of a global solution, and since any non-scalar matrix is $\mathrm{GL}_n(R)$-conjugate to one in Laffey-Reams form, our main result follows (the case when A is scalar requires a separate discussion, but is easy).

Once the main result has been established for a PID, it is easy to deduce it for an arbitrary principal ideal ring (not necessary an integral domain).

We end this introduction with a word on notation. A ring (without further specification) will mean a commutative ring with identity. Throughout, we will use 1_n to denote the identity matrix in $\mathfrak{gl}_n(S)$, where S is a ring. If $X\in\mathfrak{gl}_n(S)$, $S[X]$ will denote the unital S-algebra generated by X.

2. The criterion of Laffey and Reams

In this section, K denotes an arbitrary field. We will prove an analogue of the Laffey–Reams criterion (see [4, Section 3] and [6, Proposition 3.3]) for a matrix in $\mathfrak{sl}_n(R)$, R a *local* PID, to be a commutator of matrices in $\mathfrak{sl}_n(R)$. This criterion plays a key role in our proof of the main theorem.

We need a couple of remarks about regular elements in $\mathfrak{sl}_n(K)$. It is well known that an element $X\in\mathfrak{gl}_n(K)$ is regular if and only if

$$C_{\mathfrak{gl}_n(K)}(X)=K[X],$$

that is, if and only if the centraliser of X in $\mathfrak{gl}_n(K)$ has dimension n. In this situation, we will say that X is $\mathfrak{gl}_n(K)$-*regular*. Similarly, if $X \in \mathfrak{sl}_n(K)$ we define X to be $\mathfrak{sl}_n(K)$-*regular* if

$$\dim C_{\mathfrak{sl}_n(K)}(X) = n - 1.$$

For $X \in \mathfrak{sl}_n(K)$ it may happen that X is $\mathfrak{gl}_n(K)$-regular but not $\mathfrak{sl}_n(K)$-regular: take for example $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_n(\mathbb{F}_2)$.

The following result describes the precise relationship between the properties $\mathfrak{sl}_n$-regular and $\mathfrak{gl}_n$-regular over a field.

Lemma 2.1. *Let $X \in \mathfrak{sl}_n(K)$. Then the following holds:*

(i) If X is $\mathfrak{sl}_n(K)$-regular, then X is $\mathfrak{gl}_n(K)$-regular.
(ii) X is $\mathfrak{sl}_n(K)$-regular if and only if it is $\mathfrak{gl}_n(K)$-regular and $\operatorname{tr}(K[X]) \neq 0$.
(iii) If $\operatorname{char} K$ does not divide n, then an element X is $\mathfrak{sl}_n(K)$-regular if and only if it is $\mathfrak{gl}_n(K)$-regular.

Proof. For the first part, note that $C_{\mathfrak{sl}_n(K)}(X)$ is either equal to $C_{\mathfrak{gl}_n(K)}(X)$ or is a hypersurface in $C_{\mathfrak{gl}_n(K)}(X)$, so $C_{\mathfrak{sl}_n(K)}(X)$ has codimension at most one in $C_{\mathfrak{gl}_n(K)}(X)$. Thus X being $\mathfrak{sl}_n(K)$-regular implies that $\dim C_{\mathfrak{gl}_n(K)}(X) \leq n$. But it is well-known that the dimension of a centraliser in $\mathfrak{gl}_n(K)$ is always at least n, so X is $\mathfrak{gl}_n(K)$-regular.

For the second part, first note that $C_{\mathfrak{sl}_n(K)}(X)$ is the kernel of the trace map $\operatorname{tr} : C_{\mathfrak{gl}_n(K)}(X) \to K$. Now, if X is $\mathfrak{sl}_n(K)$-regular, then by the previous part, X is $\mathfrak{gl}_n(K)$-regular, so $C_{\mathfrak{gl}_n(K)}(X) = K[X]$. Thus $\dim C_{\mathfrak{sl}_n(K)}(X) = n-1$ implies that this trace map is surjective, that is, that $\operatorname{tr}(K[X]) \neq 0$. Conversely, if X is $\mathfrak{gl}_n(K)$-regular and $\operatorname{tr}(K[X]) \neq 0$, then $\dim C_{\mathfrak{gl}_n(K)}(X) = n$ and $\operatorname{tr} : C_{\mathfrak{gl}_n(K)}(X) \to K$ is surjective, so the kernel has dimension $n - 1$.

Finally, when $\operatorname{char} K$ does not divide n and X is $\mathfrak{gl}_n(K)$-regular, then $\operatorname{tr}(1_n) = n \neq 0$, so the previous part implies that X is $\mathfrak{sl}_n(K)$-regular. ☐

Proposition 2.2. *Let $X \in \mathfrak{sl}_n(K)$ be $\mathfrak{sl}_n(K)$-regular and let $A \in \mathfrak{sl}_n(K)$. Then $A = [X, Y]$ for some $Y \in \mathfrak{sl}_n(K)$ if and only if $\operatorname{tr}(X^r A) = 0$ for all $r = 1, \ldots, n-1$.*

Proof. Since X is $\mathfrak{gl}_n(K)$-regular by Lemma 2.1, the set $\{1_n, X, \ldots, X^{n-1}\}$ is linearly independent over K, so the subspace

$$V = \{B \in \mathfrak{sl}_n(K) \mid \operatorname{tr}(X^r B) = 0 \text{ for } r = 1, \ldots, n-1\}$$

has dimension $n^2 - n$. The kernel of the linear map $\mathfrak{sl}_n(K) \to \mathfrak{sl}_n(K)$, $Y \mapsto [X, Y]$ is equal to the centraliser $C_{\mathfrak{sl}_n(K)}(X)$, which has dimension $n-1$ since X is $\mathfrak{sl}_n(K)$-regular. Thus the image $[X, \mathfrak{sl}_n(K)]$ of the map $Y \mapsto [X, Y]$ has dimension $n^2 - n$. But if $A \in [X, \mathfrak{sl}_n(K)]$, there exists a $Y \in \mathfrak{sl}_n(K)$ such that for every $r = 1, \ldots, n-1$ we have

$$\operatorname{tr}(X^r A) = \operatorname{tr}(X^r(XY - YX)) = \operatorname{tr}(X^{r+1}Y) - \operatorname{tr}(X^r Y X) = 0.$$

Thus $[X, \mathfrak{sl}_n(K)] \subseteq V$. Since $\dim V = \dim[X, \mathfrak{sl}_n(K)]$ we conclude that $V = [X, \mathfrak{sl}_n(K)]$. ☐

If S is a ring, $I \subseteq S$ an ideal and $X \in \mathfrak{gl}_n(S)$, we denote by X_I the image of X under the canonical map $\mathfrak{gl}_n(S) \to \mathfrak{gl}_n(S/I)$.

Lemma 2.3. *Let S be a local ring (commutative, with identity) with maximal ideal $\mathfrak{m}$. Let $X \in \mathfrak{sl}_n(S)$ be such that $X_\mathfrak{m}$ is $\mathfrak{sl}_n(S/\mathfrak{m})$-regular. Then the canonical map*

$$C_{\mathfrak{sl}_n(S)}(X) \longrightarrow C_{\mathfrak{sl}_n(S/\mathfrak{m})}(X_\mathfrak{m})$$

is surjective.

Proof. As $C_{\mathfrak{sl}_n(S/\mathfrak{m})}(X_\mathfrak{m})$ has dimension $n-1$ and is the kernel of the trace map $\mathrm{tr} : C_{\mathfrak{gl}_n(S/\mathfrak{m})}(X_\mathfrak{m}) \to S/\mathfrak{m}$, this map must be surjective. Thus, there exists an $a \in C_{\mathfrak{gl}_n(S/\mathfrak{m})}(X_\mathfrak{m})$ such that $\mathrm{tr}(a) = 1$. Since $X_\mathfrak{m}$ is $\mathfrak{sl}_n(S/\mathfrak{m})$-regular, it is also $\mathfrak{gl}_n(S/\mathfrak{m})$-regular, so

$$C_{\mathfrak{gl}_n(S/\mathfrak{m})}(X_\mathfrak{m}) = (S/\mathfrak{m})[X_\mathfrak{m}].$$

Let $\hat{a} \in S[X] \subseteq C_{\mathfrak{gl}_n(S)}(X)$ be any lift of a. Then $\mathrm{tr}(\hat{a}) \in 1 + \mathfrak{m}$, so $\mathrm{tr}(\hat{a})$ is a unit in S since S is a local ring. Now, let $b \in C_{\mathfrak{sl}_n(S/\mathfrak{m})}(X_\mathfrak{m}) \subseteq (S/\mathfrak{m})[X_\mathfrak{m}]$, and choose a lift $\hat{b} \in S[X]$ of b. Then $\mathrm{tr}(\hat{b}) \in \mathfrak{m}$, so the element $\hat{b} - \mathrm{tr}(\hat{b})\,\mathrm{tr}(\hat{a})^{-1}\hat{a} \in C_{\mathfrak{sl}_n(S)}(X)$ maps onto $b \in C_{\mathfrak{sl}_n(S/\mathfrak{m})}(X_\mathfrak{m})$. □

The following result is a local version of the criterion of Laffey and Reams ([6, Proposition 3.3]), with the difference that we need $X_\mathfrak{p}$ to be $\mathfrak{sl}_n(R/\mathfrak{p})$-regular to ensure that $Y \in \mathfrak{sl}_n(R)$ rather than just in $\mathfrak{gl}_n(R)$.

Proposition 2.4. *Assume that R is a local PID with maximal ideal $\mathfrak{p}$, let $A \in \mathfrak{sl}_n(R)$ and let $X \in \mathfrak{sl}_n(R)$ be such that $X_\mathfrak{p}$ is $\mathfrak{sl}_n(R/\mathfrak{p})$-regular. Then $A = [X, Y]$ for some $Y \in \mathfrak{sl}_n(R)$ if and only if $\mathrm{tr}(X^r A) = 0$ for $r = 1, \ldots, n-1$.*

Proof. Clearly the condition $\mathrm{tr}(X^r A) = 0$ for all $r \geq 1$ is necessary for A to be of the form $[X, Y]$ with $Y \in \mathfrak{sl}_n(R)$. Conversely, suppose that $\mathrm{tr}(X^r A) = 0$ for $r = 1, \ldots, n-1$. Let F be the field of fractions of R. We claim that X is $\mathfrak{sl}_n(F)$-regular, considered as an element of $\mathfrak{sl}_n(F)$. Indeed, by [6, Proposition 2.6] X is $\mathfrak{gl}_n(F)$-regular, and since $X_\mathfrak{p}$ is $\mathfrak{sl}_n(R/\mathfrak{p})$-regular, there exists an element $a \in R[X]$ such that $\mathrm{tr}(a) \neq 0$. Thus $\mathrm{tr}(F[X]) \neq 0$, and so X is $\mathfrak{sl}_n(F)$-regular by Lemma 2.1.

Now, by Proposition 2.2 we have $A = [X, M]$ for some $M \in \mathfrak{sl}_n(F)$. Let p be a generator of $\mathfrak{p}$. Then there exists a non-negative integer m such that $p^m M \in \mathfrak{sl}_n(R)$, and we have $[X, p^m M] = p^m [X, M] = p^m A$. Choose m to be minimal with respect to the property that $[X, C] = p^m A$ for some $C \in \mathfrak{sl}_n(R)$. Assume that $m > 0$. Then $[X_\mathfrak{p}, C_\mathfrak{p}] = 0$, so $X_\mathfrak{p}$ commutes with $C_\mathfrak{p}$. Since $X_\mathfrak{p}$ is $\mathfrak{sl}_n(R/\mathfrak{p})$-regular, there exists a $\hat{C} \in C_{\mathfrak{sl}_n(R)}(X)$ such that $\hat{C}_\mathfrak{p} = C_\mathfrak{p}$, by Lemma 2.3. Thus $C = \hat{C} + pD$, for some $D \in \mathfrak{sl}_n(R)$, so

$$[X, C] = [X, pD] = p[X, D] = p^m A.$$

Cancelling a factor of p, we obtain a contradiction to the minimality of m. Thus $m = 0$, and the result is proved. □

3. The matrices $X(\mathbf{x}, a)$

Let S be a ring (commutative with identity), $n \geq 3$, $\mathbf{x} = (x_1, \ldots, x_{n-1})^\mathsf{T} \in S^{n-1}$ and $a \in S$. The key to our main result is to consider the following matrices:

$$X(\mathbf{x}, a) = \begin{pmatrix} 0 & 0 & 0 \cdots & 0 \\ x_1 & 0 & 1 \ddots & \vdots \\ \vdots & \vdots & 0 \ddots & 0 \\ \vdots & 0 & \vdots \ddots & 1 \\ x_{n-1} & a & 0 \cdots & 0 \end{pmatrix} \in \mathfrak{sl}_n(S),$$

that is, $X(\mathbf{x}, a) = (m_{ij})$, where

$$\begin{cases} m_{i,i+1} = 1 & \text{for } i = 2, \ldots, n-1, \\ m_{i1} = x_{i-1} & \text{for } i = 2, \ldots, n-2, \\ m_{n,2} = a & \\ m_{ij} = 0 & \text{otherwise.} \end{cases}$$

We can write $X(\mathbf{x}, a)$ in block form as

$$X(\mathbf{x}, a) = \begin{pmatrix} 0 & \overline{0} \\ \mathbf{x} & P \end{pmatrix},$$

where $\overline{0} = (0, \ldots, 0)$ is a $1 \times n$ matrix and $P = (p_{ij})$, $1 \leq i, j \leq n-1$, where $p_{i,i+1} = 1$ for $i = 1, \ldots, n-2$, $p_{n-1,1} = a$ and $p_{ij} = 0$ otherwise. Thus, P is the (row-wise) companion matrix of the polynomial $x^{n-1} - a$.

Lemma 3.1. *Let $P \in \mathfrak{sl}_{n-1}(S)$ be as above, and let $\mathbf{y} = (y_1, \ldots, y_{n-1})^\mathsf{T} \in S^{n-1}$. Then, for any $z \in S$, and $r = 1, \ldots, n-1$, we have*

$$\operatorname{tr}(P^{r-1}\mathbf{y}(z, 0, \ldots, 0)) = zy_r.$$

Proof. Write $P^{r-1} = (p_{ij}^{(r-1)})$, for $1 \leq i, j \leq n-1$. Since each column in $\mathbf{y}(z, 0, \ldots, 0)$, except for the first one, is zero, we have

$$\operatorname{tr}(P^{r-1}\mathbf{y}(z, 0, \ldots, 0)) = (p_{11}^{(r-1)}, p_{12}^{(r-1)}, \ldots, p_{1,n-1}^{(r-1)})z\mathbf{y}.$$

Since P is a companion matrix, there exists a $v \in S^{n-1}$ such that $\{v, Pv, \ldots, P^{n-2}v\}$ is an S-basis for S^{n-1} and P is the matrix of the linear map defined by P with respect to this basis. Thus, for each $r = 1, \ldots, n-1$, the first row of P^{r-1} is $(p_{11}^{(r-1)}, p_{12}^{(r-1)}, \ldots, p_{1,n-1}^{(r-1)})$, where $p_{1r}^{(r-1)} = 1$ and all other $p_{1j} = 0$. Hence

$$(p_{11}^{(r-1)}, p_{12}^{(r-1)}, \ldots, p_{1,n-1}^{(r-1)})z\mathbf{y} = zy_r,$$

and the lemma follows. □

Lemma 3.2. *For $r = 1, \ldots, n-1$ we have*

$$X(\mathbf{x}, a)^r = \begin{pmatrix} 0 & \overline{0} \\ P^{r-1}\mathbf{x} & P^r \end{pmatrix},$$

In particular, $\operatorname{tr}(X(\mathbf{x}, a)^r) = 0$ *for* $r = 1, \ldots, n-2$, *and* $\operatorname{tr}(X(\mathbf{x}, a)^{n-1}) = (n-1)a$.

Proof. The expression for $X(\mathbf{x},a)^r$ follows easily, using block-multiplication of matrices. The assertion about the trace of $X(\mathbf{x},a)^r$ for $r=1,\dots,n-2$ follows from a simple induction argument, proving that for each $r=1,\dots,n-2$, we have $P^r=(p_{ij}^{(r)})$, where $p_{i,i+r}^{(r)}=1$ for $i=1,\dots,n-1-r$ and $p_{n-1-r+j,j}^{(r)}=a$ for $j=1,\dots,r$, and $p_{ij}^{(r)}=0$ otherwise. Finally, the relation $\operatorname{tr}(X(\mathbf{x},a)^{n-1})=(n-1)a$ follows from the fact that the characteristic polynomial of P is $x^{n-1}-a$. $\square$

Lemma 3.3. *Let K be a field, $x_1,\dots,x_{n-1}\in K^{n-1}$ and $a\in K$. If either $x_{n-1}\neq 0$ or $a\neq 0$, then $X(\mathbf{x},a)$ is $\mathfrak{gl}_n(K)$-regular. If $a\neq 0$, then $X(\mathbf{x},a)$ is $\mathfrak{sl}_n(K)$-regular.*

Proof. For simplicity, write $X=X(\mathbf{x},a)$. We will show that if $x_{n-1}\neq 0$ or $a\neq 0$, then X is $\mathfrak{gl}_n(K)$-regular, by showing that $\{1_n,X,\dots,X^{n-1}\}$ is linearly independent. Lemma 3.2 implies that $\{1_n,X,\dots,X^{n-2}\}$ is linearly independent because P is regular, so $\{1_{n-1},P,\dots,P^{n-2}\}$ is linearly independent. Moreover, by Lemma 3.2 and its proof, we have

$$X^{n-1}=\begin{pmatrix}0 & \overline{0}\\ P^{n-2}\mathbf{x} & a1_{n-1}\end{pmatrix},\qquad\text{where}\qquad P^{n-2}\mathbf{x}=\begin{pmatrix}x_{n-1}\\ ax_1\\ \vdots\\ ax_{n-2}\end{pmatrix}.$$

Thus, since P^i has zero diagonal for all $r=1,\dots,n-2$ (see the proof of Lemma 3.2), we conclude that X^{n-1} is not a linear combination of $1_n,X,\dots,X^{n-2}$ if $a\neq 0$. On the other hand, if $a=0$ and $x_{n-1}\neq 0$, then X^{n-1} is the matrix whose $(2,1)$-entry is x_{n-1} and all other entries are zero. Since each matrix in $\{1_n,X,\dots,X^{n-2}\}$ has a non-zero (i,j)-entry for some $(i,j)\neq(2,1)$, we conclude that X^{n-1} is not a linear combination of $1_n,X,\dots,X^{n-2}$ if $a=0$ and $x_{n-1}\neq 0$.

Suppose now that $a\neq 0$; then X is $\mathfrak{gl}_n(K)$-regular. If $\operatorname{char}K\nmid n$, Lemma 2.1 implies that X is $\mathfrak{sl}_n(K)$-regular. On the other hand, if $\operatorname{char}K\mid n$, then

$$\operatorname{tr}(X^{n-1})=(n-1)a=-a,$$

by Lemma 3.2, so $\operatorname{tr}(K[X])\neq 0$ and Lemma 2.1 implies that X is $\mathfrak{sl}_n(K)$-regular. $\square$

4. The field case

In this section we give a proof of our main result in the case where $R=K$ is a field. We give a separate proof in this case, as it is simpler than for a general PID. The result over a field was first proved by Thompson [7], who also showed that, apart for some small exceptions, one of the matrices X can in fact be taken to be nilpotent. We give a new proof of Thompson's result, but instead of showing that X can be chosen to be nilpotent, we show that it can be taken to be $\mathfrak{gl}_n(K)$-regular (and often $\mathfrak{sl}_n(K)$-regular).

First let $n=2$. For $x,y,z,s,t,u\in K$ we have

$$\left[\begin{pmatrix}x & y\\ z & -x\end{pmatrix},\begin{pmatrix}s & t\\ u & -s\end{pmatrix}\right]=\begin{pmatrix}uy-tz & 2(tx-sy)\\ 2(sz-ux) & tz-uy\end{pmatrix}.$$

Thus, if $\operatorname{char} K = 2$, a matrix in $\mathfrak{sl}_2(K)$ is of the form $[X, Y]$ for $X, Y \in \mathfrak{sl}_2(K)$ if and only if it is scalar. On the other hand, if $\operatorname{char} K \neq 2$ and $a, b, c \in K$, then

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{cases} \left[\left(\begin{smallmatrix} 0 & 1 \\ -\frac{c}{b} & 0 \end{smallmatrix}\right), \begin{pmatrix} -\frac{b}{2} & 0 \\ a & \frac{b}{2} \end{pmatrix} \right] & \text{if } b \neq 0, \\ \left[\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} \frac{c}{2} & -a \\ 0 & -\frac{c}{2} \end{pmatrix} \right] & \text{if } b = 0. \end{cases}$$

Note that all of the matrices involved in the above commutators are $\mathfrak{gl}_n(K)$-regular.

Lemma 4.1. *Let S be a ring (commutative with identity) such that $n = 1 + \cdots + 1 = 0$ in S. Then, for every $\lambda \in S$ there exist $X, Y \in \mathfrak{sl}_n(S)$ such that X is $\mathfrak{gl}_n(S)$-regular and $[X, Y] = \lambda 1_n$.*

Proof. Take $X = (x_{ij})$, where $x_{i,i+1} = 1$ for $i = 1, \ldots, n-1$ and $x_{ij} = 0$ otherwise, and $Y = (y_{ij})$, where $y_{j+1,j} = j$, for $j = 1, \ldots, n-1$ and $y_{ij} = 0$ otherwise. Then X is a companion matrix, hence regular as an element of $\mathfrak{gl}_n(S)$. A direct computation shows that $[X, Y] = 1_n$, because $-(n-1) = 1$ in S, and thus $[X, \lambda Y] = \lambda 1_n$. □

Remark 4.2. If $S = K$ is a field, Lemma 4.1 does not hold if X is required to be $\mathfrak{sl}_n(K)$-regular; in fact, the X in the lemma is necessarily not $\mathfrak{sl}_n(K)$-regular, unless $\lambda = 0$. The author was alerted to the following simple argument by a referee: Suppose that $[X, Y] = \lambda 1_n$ where $\lambda \neq 0$ and X is $\mathfrak{gl}_n(K)$-regular. Then $\operatorname{tr}(X^i \lambda 1_n) = \lambda \operatorname{tr}(X^i) = 0$, hence $\operatorname{tr}(X^i) = 0$, for all $i = 0, \ldots, n-1$. Thus X is not $\mathfrak{sl}_n(K)$-regular, by Lemma 2.1.

Theorem 4.3. *Let K be a field and $A \in \mathfrak{sl}_n(K)$, with $n \geq 3$. Then there exist $X, Y \in \mathfrak{sl}_n(K)$ such that $[X, Y] = A$. Moreover, if A is scalar, X can be chosen to be $\mathfrak{gl}_n(K)$-regular and if A is non-scalar, X can be chosen to be $\mathfrak{sl}_n(K)$-regular.*

Proof. Assume first that A is scalar. Then either $A = 0$ or $\operatorname{char} K$ divides n. The former case is trivial, and the latter follows from Lemma 4.1.

Assume now that A is not scalar and let $A = (a_{ij})$. Then the rational canonical form implies that after a possible $\mathrm{GL}_n(K)$-conjugation, we can assume that $a_{11} = 0$, $a_{12} = 1$ and $a_{ij} = 0$ whenever $j \geq i + 2$. We will show that $x_1, \ldots, x_{n-1} \in K$ can be chosen such that $\operatorname{tr}(X(\mathbf{x}, 1)^r A) = 0$ for each $r = 1, \ldots, n-1$. By Lemma 3.2 we have

$$X(\mathbf{x}, 1)^r = \begin{pmatrix} 0 & \bar{0} \\ P^{r-1}\mathbf{x} & P^r \end{pmatrix},$$

where $P = (p_{ij})$, $1 \leq i, j, \leq n-1$ is such that $p_{i,i+1} = 1$ for $i = 1, \ldots, n-2$, $p_{n-1,1} = 1$ and $p_{ij} = 0$ otherwise. Writing A in block-form, we have

$$A = \begin{pmatrix} 0 & (1, 0, \ldots, 0) \\ \mathbf{a} & Q \end{pmatrix},$$

where $\mathbf{a}$ is an $n \times 1$ matrix and $Q \in \mathfrak{gl}_{n-1}(K)$. Thus

$$X(\mathbf{x}, 1)^r A = \begin{pmatrix} 0 & \bar{0} \\ P^r \mathbf{a} & Q' \end{pmatrix},$$

where $Q' = P^{r-1}\mathbf{x}(1, 0, \ldots, 0) + P^r Q$. Thus, by Lemma 3.1,

$$\operatorname{tr}(X(\mathbf{x}, 1)^r A) = \operatorname{tr}(Q') = x_r + \operatorname{tr}(P^r Q),$$

for each $r = 1, \dots, n-1$. Put $x_r = -\operatorname{tr}(P^r Q)$, so that $\operatorname{tr}(X(\mathbf{x}, 1)^r A) = 0$, for $r = 1, \dots, n-1$. By Lemma 3.3 $X(\mathbf{x}, 1)$ is $\mathfrak{sl}_n(K)$-regular, so Proposition 2.2 implies that there exists a $Y \in \mathfrak{sl}_n(K)$ such that

$$[X(\mathbf{x}, 1), Y] = A.$$

□

Remark 4.4. Our approach cannot be modified to yield Thompson's result that X can be taken to be nilpotent. The reason for this is that $X(\mathbf{x}, a)$ is nilpotent if and only if P is nilpotent if and only if $a = 0$. Therefore, even if $X(\mathbf{x}, a)$ is nilpotent and $\mathfrak{gl}_n(K)$-regular, it cannot be $\mathfrak{sl}_n(K)$-regular, because $\operatorname{tr}(X(\mathbf{x}, 0)^r) = 0$ for every $r = 1, \dots, n-1$.

5. Proof of the Main Theorem

Throughout this section, R is an arbitrary PID. Note that we consider fields as special types of PIDs.

Before proving our main result (Theorem 5.3 below), we give a new and simplified proof of the main result in [6] that any $A \in \mathfrak{sl}_n(R)$ is a commutator of matrices in $\mathfrak{gl}_n(R)$. The proof of our main result is a bit harder, as it involves a special analysis for certain prime ideals. Both proofs make essential use of the Laffey-Reams form and rely on the following key result:

Lemma 5.1. *Suppose that $A = (a_{ij}) \in \mathfrak{sl}_n(R)$ is in Laffey-Reams form, that is, $a_{ij} = 0$ for $j \geq i+2$ and $A \equiv a_{11} 1_n \mod (a_{12})$. Then there exists an $\mathbf{x} = (x_1, \dots, x_{n-1})^{\mathsf{T}} \in R^{n-1}$, with $x_{n-1} = a_{11}$, such that*

$$\operatorname{tr}(X(\mathbf{x}, a_{12})^r A) = 0,$$

for each $r = 1, \dots, n-1$.

Proof. By Lemma 3.2 we have

$$X(\mathbf{x}, a_{12})^r = \begin{pmatrix} 0 & \bar{0} \\ P^{r-1}\mathbf{x} & P^r \end{pmatrix},$$

where $P = (p_{ij})$, $1 \leq i, j, \leq n-1$ is such that $p_{i,i+1} = 1$ for $i = 1, \dots, n-2$, $p_{n-1,1} = a_{12}$ and $p_{ij} = 0$ otherwise (i.e., P is the row-wise companion matrix of $x^{n-1} - a_{12}$). Writing A in block-form, we have

$$A = \begin{pmatrix} a_{11} & (a_{12}, 0, \dots, 0) \\ \mathbf{a} & Q \end{pmatrix},$$

where $\mathbf{a}$ is an $n \times 1$ matrix and $Q \in \mathfrak{gl}_{n-1}(R)$. Thus

$$X(\mathbf{x}, a_{12})^r A = \begin{pmatrix} 0 & \bar{0} \\ a_{11} P^{r-1}\mathbf{x} + P^r \mathbf{a} & Q' \end{pmatrix},$$

where $Q' = P^{r-1}\mathbf{x}(a_{12}, 0, \dots, 0) + P^r Q$. Thus, by Lemma 3.1,

$$\operatorname{tr}(X(\mathbf{x}, a_{12})^r A) = \operatorname{tr}(Q') = a_{12} x_r + \operatorname{tr}(P^r Q),$$

for each $r = 1, \dots, n-1$. We have $\operatorname{tr}(P^r) \equiv 0 \mod (a_{12})$, for $r = 1, \dots, n-1$, and since $A \equiv a_{11} 1_n \mod (a_{12})$ it follows that $Q \equiv a_{11} 1_{n-1} \mod (a_{12})$. Thus

$$\operatorname{tr}(P^r Q) \equiv a_{11} \operatorname{tr}(P^r) \equiv 0 \mod (a_{12}),$$

so there exist $m_r \in R$ such that $\operatorname{tr}(P^r Q) = a_{12}m_r$, for each $r = 1, \ldots, n-1$. Put $x_r = -m_r$, so that

$$\operatorname{tr}(X(\mathbf{x}, a_{12})^r A) = 0,$$

for $r = 1, \ldots, n-1$.

Finally, we claim that $\operatorname{tr}(P^{n-1}Q) = -a_{11}a_{12}$, so that

$$x_{n-1} = a_{11}.$$

Indeed, since P has characteristic polynomial $x^{n-1} - a_{12}$, we have $P^{n-1} = a_{12}1_{n-1}$, so $\operatorname{tr}(P^{n-1}Q) = a_{12}\operatorname{tr}(Q) = a_{12}(-a_{11})$, as claimed. □

The following result is essentially [6, Theorem 6.3], but the result here is stronger in that it says that X can be taken in $\mathfrak{sl}_n(R)$ and such that it is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular mod any maximal ideal $\mathfrak{p}$ of R.

Theorem 5.2. *Let $A \in \mathfrak{sl}_n(R)$ with $n \geq 2$. Then there exist matrices $X \in \mathfrak{sl}_n(R)$ and $Y \in \mathfrak{gl}_n(R)$ such that $[X, Y] = A$, where X can be chosen such that $X_{\mathfrak{p}}$ is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular for every maximal ideal $\mathfrak{p}$ of R.*

Proof. For $n = 2$ this is proved separately (see the proof of [6, Theorem 6.3]). Assume from now on that $n \geq 3$. First, if A is scalar, then $A \in \mathfrak{sl}_n(R)$ implies that either $A = 0$ or $n = 0$ in R. The former case is trivial, while the latter follows from Lemma 4.1.

Assume now that A is not scalar and let $A = (a_{ij})$. After a possible $\mathrm{GL}_n(R)$-conjugation, we can assume that A is in Laffey–Reams form; see [6, Theorem 5.6]. Moreover, we may assume that $(a_{11}, a_{12}) = (1)$, because if d is a common divisor of a_{11} and a_{12}, we can write $A = dA'$ for A' in Laffey–Reams form and if $A' = [X, Y]$ with X, Y as in the theorem, then $A = [X, dY]$.

By Lemma 5.1, there exists an $\mathbf{x} = (x_1, \ldots, x_{n-1})^\intercal \in R^{n-1}$, with $x_{n-1} = a_{11}$, such that

$$\operatorname{tr}(X(\mathbf{x}, a_{12})^r A) = 0,$$

for each $r = 1, \ldots, n-1$. Since $x_{n-1} = a_{11}$ and $(a_{11}, a_{12}) = (1)$, we have, for every maximal ideal $\mathfrak{p}$ of R, that either $x_{n-1} \notin \mathfrak{p}$ or $a_{12} \notin \mathfrak{p}$, and therefore $X_{\mathfrak{p}}$ is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular, by Lemma 3.3. Thus, by [6, Proposition 3.3], there exists a $Y \in \mathfrak{gl}_n(R)$ such that

$$[X(\mathbf{x}, a_{12}), Y] = A.$$

□

We now come to the proof of our main theorem. Just like the proof of the above theorem, our proof uses Lemma 5.1, but since here $X(\mathbf{x}, a_{12})_{\mathfrak{p}}$ cannot in general be $\mathfrak{sl}_n(R/\mathfrak{p})$-regular for all maximal ideals (cf. Remark 4.2), we need to treat the exceptional primes separately, and this requires us to pass to the localisations $R_{\mathfrak{p}}$, for various prime ideals $\mathfrak{p} \in \operatorname{Spec}(R)$. For an element $X \in \mathfrak{gl}_n(R)$ we will write $X(\mathfrak{p})$ for its canonical image in $\mathfrak{gl}_n(R_{\mathfrak{p}})$, not to be confused with $X_{\mathfrak{p}} \in \mathfrak{gl}_n(R/\mathfrak{p})$. For any element $x \in R$, we will use the same symbol x to denote the image of x under the canonical injection $R \hookrightarrow R_{\mathfrak{p}}$, and the context will make it clear in which ring we are working. Similarly, we will denote the maximal ideal of $R_{\mathfrak{p}}$ by $\mathfrak{p}$ and will identify $X_{\mathfrak{p}} \in \mathfrak{gl}_n(R/\mathfrak{p})$ with the image of $X(\mathfrak{p})$ in $\mathfrak{gl}_n(R_{\mathfrak{p}}/\mathfrak{p})$.

We will prove that for fixed $A, X \in \mathfrak{sl}_n(R)$, and for any maximal ideal $\mathfrak{p}$ of R, there exists a solution $Y(\mathfrak{p}) \in \mathfrak{sl}_n(R_{\mathfrak{p}})$ to the localised equation $[X(\mathfrak{p}), Y(\mathfrak{p})] = A(\mathfrak{p})$. Since the equations $[X, Y] = A$, $\operatorname{tr}(Y) = 0$ in Y are equivalent to a system of linear equations in the entries of Y, the well known (and easy to prove) local-global principle for systems of linear equations (see, e.g., [3, Proposition 1]) implies the existence of a global solution.

Theorem 5.3. *Let $A \in \mathfrak{sl}_n(R)$ for $n \geq 3$. Then there exist matrices $X, Y \in \mathfrak{sl}_n(R)$ such that $[X, Y] = A$, where X can be chosen such that $X_\mathfrak{p}$ is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular for every maximal ideal $\mathfrak{p}$ of R. Moreover, X can be chosen such that $X_\mathfrak{p}$ is $\mathfrak{sl}_n(R/\mathfrak{p})$-regular for every $\mathfrak{p}$ such that $A_\mathfrak{p}$ is not scalar.*

Proof. Assume first that A is scalar. Then $A \in \mathfrak{sl}_n(R)$ implies that either $A = 0$ or $n = 0$ in R. The former case is trivial, while the latter follows from Lemma 4.1.

Assume from now on that A is not scalar and let $A = (a_{ij})$. After a possible $\mathrm{GL}_n(R)$-conjugation, we can assume that A is in Laffey–Reams form. Moreover, we may assume that $(a_{11}, a_{12}) = (1)$, because if d is a common divisor of a_{11} and a_{12}, we can write $A = dA'$ for A' in Laffey–Reams form, and if A' is a commutator of two matrices in $\mathfrak{sl}_n(R)$, then so is A.

By Lemma 5.1, there exists an $\mathbf{x} = (x_1, \ldots, x_{n-1})^\mathsf{T} \in R^{n-1}$, with $x_{n-1} = a_{11}$, such that

$$\operatorname{tr}(X(\mathbf{x}, a_{12})^r A) = 0,$$

for each $r = 1, \ldots, n-1$. From now on, let $X := X(\mathbf{x}, a_{12})$. Since $(a_{11}, a_{12}) = (1)$, we have, for every maximal ideal $\mathfrak{p}$ of R, that either $x_{n-1} \notin \mathfrak{p}$ or $a_{12} \notin \mathfrak{p}$, and therefore that $X_\mathfrak{p}$ is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular; see Lemma 3.3. Moreover, since A is in Laffey-Reams form, we have $A \equiv a_{11} 1_n \bmod (a_{12})$, and this, combined with the fact that $\operatorname{tr}(A) = 0$ and $(a_{11}, a_{12}) = (1)$, implies that

$$n \in (a_{12}). \tag{5.1}$$

We will now pass to the localisations $R_\mathfrak{p}$ for various maximal ideals $\mathfrak{p}$ of R. Let $\mathfrak{p}$ be any maximal ideal of R. Then we have the local relations

$$\operatorname{tr}(X(\mathfrak{p})^r A(\mathfrak{p})) = 0, \qquad r = 1, \ldots, n-1.$$

in $R_\mathfrak{p}$. First, suppose that $A_\mathfrak{p}$ is not scalar. Then $a_{12} \notin \mathfrak{p}$, so the matrix $X(\mathfrak{p})_\mathfrak{p} = X_\mathfrak{p}$ is $\mathfrak{sl}_n(R_\mathfrak{p}/\mathfrak{p})$-regular, by Lemma 3.3, and so, by Proposition 2.4, there exists a $Y(\mathfrak{p}) \in \mathfrak{sl}_n(R_\mathfrak{p})$ such that

$$[X(\mathfrak{p}), Y(\mathfrak{p})] = A(\mathfrak{p}).$$

Next, suppose that $A_\mathfrak{p}$ is scalar, so that $a_{12} \in \mathfrak{p}$. Let F be the field of fractions of R. Since A is not scalar, we have $a_{12} \neq 0$, so X is $\mathfrak{sl}_n(F)$-regular as an element of $\mathfrak{sl}_n(F)$, by Lemma 3.3. Hence, there exists a $Y(0) \in \mathfrak{sl}_n(F)$ such that $[X, Y(0)] = A$. Clearing denominators in $Y(0)$ and passing to the localisation at $\mathfrak{p}$, we conclude that there exists a power p^m of a generator $p \in R_\mathfrak{p}$ of $\mathfrak{p}$ and a $Q \in \mathfrak{sl}_n(R_\mathfrak{p})$, such that

$$[X(\mathfrak{p}), Q] = p^m A(\mathfrak{p}). \tag{5.2}$$

Let $m \geq 0$ be the minimal integer such that (5.2) holds for some $Q \in \mathfrak{sl}_n(R_\mathfrak{p})$. We will show that $m = 0$. For a contradiction, assume that $m \geq 1$. Reducing (5.2) mod $\mathfrak{p}$, we obtain $[X_\mathfrak{p}, Q_\mathfrak{p}] = 0$, so $Q_\mathfrak{p}$ commutes with $X_\mathfrak{p}$. Since $X_\mathfrak{p}$ is $\mathfrak{gl}_n(R/\mathfrak{p})$-regular,

$$Q = f(X(\mathfrak{p})) + pD,$$

for some polynomial $f(T) \in R_{\mathfrak{p}}[T]$ of degree at most $n-1$ and some $D \in \mathfrak{gl}_n(R_{\mathfrak{p}})$. Write $f(T) = c_0 + c_1 T + \cdots + c_{n-1}T^{n-1}$, for $c_i \in R_{\mathfrak{p}}$. By Lemma 3.2, we have

$$\operatorname{tr}(X^i) = \begin{cases} n & \text{if } i = 0, \\ (n-1)a_{12} & \text{if } i = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

which implies

$$\operatorname{tr}(X(\mathfrak{p})^i) = \begin{cases} n & \text{if } i = 0, \\ (n-1)a_{12} & \text{if } i = n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Hence

$$0 = \operatorname{tr}(Q) = \sum_{i=0}^{n-1} c_i \operatorname{tr}(X(\mathfrak{p})^i) + p\operatorname{tr}(D) = c_0 n + c_{n-1}(n-1)a_{12} + p\operatorname{tr}(D). \tag{5.4}$$

Moreover, we have $[X(\mathfrak{p}), Q] = [X(\mathfrak{p}), pD] = p^m A(\mathfrak{p})$, so

$$0 = \operatorname{tr}(pDp^m A(\mathfrak{p})) = p^{m+1}\operatorname{tr}(DA(\mathfrak{p})),$$

and thus $\operatorname{tr}(DA(\mathfrak{p})) = 0$. Since $A(\mathfrak{p}) \equiv a_{11}1_n \bmod (a_{12})$ and $(a_{11}, a_{12}) = (1)$, we conclude that

$$\operatorname{tr}(D) \in (a_{12}). \tag{5.5}$$

Since $n \in (a_{12})$ by (5.1), we have $n = a_{12}n'$ for some $n' \in R_{\mathfrak{p}}$. Moreover, since $a_{12} \in \mathfrak{p}$ and $R_{\mathfrak{p}}$ is a local ring, $n-1$ is a unit in $R_{\mathfrak{p}}$, so we can define the matrix

$$Q' = (c_0 n'(n-1)^{-1} + c_{n-1})X(\mathfrak{p})^{n-1} + pD.$$

By (5.3) and (5.4) we have

$$\operatorname{tr}(Q') = c_0 n + c_{n-1}(n-1)a_{12} + p\operatorname{tr}(D) = \operatorname{tr}(Q) = 0.$$

By (5.5) this implies that $c_0 n + c_{n-1}(n-1)a_{12} \in (pa_{12})$, and thus

$$c_0 n'(n-1)^{-1} + c_{n-1} \in (p).$$

Writing $c_0 n'(n-1)^{-1} + c_{n-1} = p\alpha$ for some $\alpha \in R_{\mathfrak{p}}$, we then get

$$[X(\mathfrak{p}), Q] = [X(\mathfrak{p}), pD] = [X(\mathfrak{p}), Q'] = p[X(\mathfrak{p}), \alpha X(\mathfrak{p})^{n-1} + D] = p^m A(\mathfrak{p}),$$

where $\operatorname{tr}(\alpha X(\mathfrak{p})^{n-1} + D) = 0$ because

$$p\operatorname{tr}(\alpha X(\mathfrak{p})^{n-1} + D) = \operatorname{tr}((c_0 n'(n-1)^{-1} + c_{n-1})X(\mathfrak{p})^{n-1} + pD) = \operatorname{tr}(Q') = 0.$$

By cancelling a factor of p, we obtain

$$[X(\mathfrak{p}), \alpha X(\mathfrak{p})^{n-1} + D] = p^{m-1}A(\mathfrak{p}),$$

which contradicts the minimality of m in (5.2). Thus $m = 0$, so there exists a $Y(\mathfrak{p}) \in \mathfrak{sl}_n(R_{\mathfrak{p}})$ such that $[X(\mathfrak{p}), Y(\mathfrak{p})] = A(\mathfrak{p})$.

We have thus proved that for any maximal ideal $\mathfrak{p}$ of R, there exists a $Y(\mathfrak{p}) \in \mathfrak{sl}_n(R_{\mathfrak{p}})$ such that

$$[X(\mathfrak{p}), Y(\mathfrak{p})] = A(\mathfrak{p}).$$

We have shown that there is a local solution $Y(\mathfrak{p})$ for every maximal ideal $\mathfrak{p}$ of R. Thus, by the local-global principle for systems of linear equations (see, e.g., [3, Proposition 1]), there exists a $Y \in \mathfrak{sl}_n(R)$ such that

$$[X, Y] = A.$$

□

In the same way as in [6, Corollary 6.4], Theorem 5.3 implies the analogous statement over any principal ideal ring (PIR), thanks to a theorem of Hungerford that any PIR is a finite product of homomorphic images of PIDs.

References

[1] A. A. Albert and Benjamin Muckenhoupt. On matrices of trace zero. *Michigan Math. J.*, 4:1–3, 1957.

[2] Clément de Seguins Pazzis. Commutators from a hyperplane of matrices. *Electron. J. Linear Algebra*, 27:39–54, 2014.

[3] J. A. Hermida and T. Sánchez-Giralda. Linear equations over commutative rings and determinantal ideals. *J. Algebra*, 99(1):72–79, 1986.

[4] Thomas J. Laffey and Robert Reams. Integral similarity and commutators of integral matrices. *Linear Algebra Appl.*, 197/198:671–689, 1994.

[5] David Lissner. Matrices over polynomial rings. *Trans. Amer. Math. Soc.*, 98:285–305, 1961.

[6] Alexander Stasinski. Similarity and commutators of matrices over principal ideal rings. *Trans. Amer. Math. Soc.*, 368:2333–2354, 2016.

[7] R. C. Thompson. Matrices with zero trace. *Israel J. Math.*, 4:33–42, 1966.

[8] L. N. Vaserstein. Noncommutative number theory. In *Algebraic K-theory and Algebraic Number Theory*, volume 83 of *Contemp. Math.*, pages 445–449. AMS, Providence, RI, 1989.

Department of Mathematical Sciences, Durham University, South Rd, Durham, DH1 3LE, UK

E-mail address: `alexander.stasinski@durham.ac.uk`